Remarks on the stochastic transport equation with Hölder drift

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Abstract

We consider a stochastic linear transport equation with a globally Hölder continuous and bounded vector field. Opposite to what happens in the deterministic case where shocks may appear, we show that the unique solution starting with a C^1-initial condition remains of class C^1 in space. We also improve some results of [8] about well-posedness. Moreover, we prove a stability property for the solution with respect to the initial datum.

1 Introduction

The aim of this paper is twofold. On one side, we review ideas and recent results about the regularization by noise in ODEs and PDEs (Section 1). On the other, we give detailed proof of two new results of regularization by noise, for linear trasport equations, related to those of the paper [8] (Theorem 7 and the results of section 4).

1.1 The ODE case

A well known but still always surprising fact is the regularization produced by noise on ordinary differential equations (ODEs). Consider the ODE in $\mathbb{R}^d$

$$\frac{d}{dt}X(t) = b(t, X(t)), \quad X(0) = x_0 \in \mathbb{R}^d$$

with $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$. If b is Lipschitz continuous and has linear growth, uniformly in t, then there exists a unique solution $X \in C([0,T];\mathbb{R}^d)$. But when b is less regular there are well-known counterexamples, like the case $d = 1$, $b(x) = 2sign(x)\sqrt{|x|}$, $x_0 = 0$ where the Cauchy problem has infinitely many solutions: $X(t) = 0$, $X(t) = t^2$, $X(t) = -t^2$, and others. The function b of this example is Hölder continuous.

Consider now the stochastic differential equation (SDE)

$$dX(t) = b(t, X(t))\,dt + \sigma dW(t), \quad X(0) = x_0 \in \mathbb{R}^d \tag{1}$$

with $\sigma \in \mathbb{R}$ and $\{W(t)\}_{t\geq 0}$ a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. We say that a continuous stochastic process $X(t,\omega)$, $t \geq 0$, $\omega \in \Omega$, adapted to the filtration $\{\mathcal{F}_t^W\}_{t\geq 0}$ of the Brownian motion, is a solution if it satisfies the identity

$$X(t,\omega) = x_0 + \int_0^t b(s, X(s,\omega))\,ds + \sigma W(t,\omega), \quad t \geq 0,$$

for P-a.e. $\omega \in \Omega$. In the Lipschitz case we have again existence and uniqueness of solutions. But now, we have more: if $\sigma \neq 0$ and $b \in L^\infty\left([0,T] \times \mathbb{R}^d; \mathbb{R}^d\right)$ then there is existence and uniqueness of solutions, [19]. The result is true even when $b \in L^q\left(0,T; L^p\left(\mathbb{R}^d; \mathbb{R}^d\right)\right)$ with $\frac{d}{p} + \frac{2}{q} < 1$, $p, q \geq 2$ [14] (the assumptions can be properly localized). Recently, we have proved in [8] the following additional result, which will be used below (the function spaces are defined in Section 1.4).

Theorem 1 *If $\sigma \neq 0$ and $b \in L^\infty\left(0,T; C_b^\alpha\left(\mathbb{R}^d; \mathbb{R}^d\right)\right)$, $\alpha \in (0,1)$, then there exists a stochastic flow of diffeomorphisms $\phi_t = \phi(t,\omega)$ associated to the SDE, with $D\phi(t,\omega)$ and $D\phi^{-1}(t,\omega)$ of class $C^{\alpha'}$ for every $\alpha' \in (0,\alpha)$.*

By stochastic flow of diffeomorphisms we mean a family of maps $\phi(t,\omega)$: $\mathbb{R}^d \to \mathbb{R}^d$ such that:

i) $\phi(t,\omega)(x_0)$ is the unique solution of the SDE for every $x_0 \in \mathbb{R}^d$;

ii) $\phi(t,\omega)$ is a diffeomorphisms of $\mathbb{R}^d$.

For several results on stochastic flows under more regular conditions on b see [15]. Let us give an idea of the proof assuming $\sigma = 1$. Introduce the vector valued non homogeneous backward parabolic equation

$$\frac{\partial U}{\partial t} + b \cdot \nabla U + \frac{1}{2}\Delta U = -b + \lambda U \qquad \text{on } [0,T]$$
$$U(T,x) = 0$$

with $\lambda \geq 0$. By parabolic regularity theory we have the following result (cf. Theorem 2 in [8]):

Theorem 2 *If $b \in L^\infty\left(0,T; C_b^\alpha\left(\mathbb{R}^d; \mathbb{R}^d\right)\right)$, $\alpha \in (0,1)$, then there exists a unique bounded and locally Lipschitz solution U with the property*

$$\frac{\partial U}{\partial t} \in L^\infty\left(0,T; C_b^\alpha(\mathbb{R}^d; \mathbb{R}^d)\right), \quad D^2 U \in L^\infty\left(0,T; C_b^\alpha\left(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d\right)\right).$$

Moreover, for large λ one has, for any $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$|\nabla U(t,x)| \leq \frac{1}{2}.$$

If $X(t)$ is a solution of the SDE, we apply Itô formula to $U(t, X(t))$ and get

$$U(t, X(t)) = U(0, x_0) + \int_0^t \mathcal{L}U(s, X(s))\, ds + \int_0^t \nabla U(s, X(s))\, dW(s)$$

where $\mathcal{L}U = \frac{\partial U}{\partial t} + b \cdot \nabla U + \frac{1}{2}\Delta U$. Hence, being $\mathcal{L}U = -b + \lambda U$,

$$U(t, X(t)) = U(0, x_0) + \int_0^t (-b + \lambda U)(s, X(s))\, ds + \int_0^t \nabla U(s, X(s))\, dW(s)$$

and thus

$$\int_0^t b(s, X(s))\, ds = U(0, x_0) - U(t, X(t)) + \int_0^t \lambda U(s, X(s))\, ds + \int_0^t \nabla U(s, X(s))\, dW(s).$$

In other words, we may rewrite the SDE as

$$X(t) = x_0 + U(0, x_0) - U(t, X(t)) + \int_0^t \lambda U(s, X(s))\, ds + \int_0^t \nabla U(s, X(s))\, dW(s) + W(t).$$

The advantage is that U is twice more regular than b and ∇U is once more regular. All terms in this equation are at least Lipschitz continuous.

From the new equation satisfied by $X(t)$ it is easy to prove uniqueness, for instance. But, arguing a little bit formally, it is also clear that we have differentiability of $X(t)$ with respect to the initial condition x_0. Indeed, if $D_h X(t)$ denotes the derivative in the direction h, we (formally) have

$$\begin{aligned} D_h X(t) = h &+ D_h U(0, x_0) - \nabla U(t, X(t))\, D_h X(t) \\ &+ \int_0^t \lambda \nabla U(s, X(s))\, D_h X(s)\, ds \\ &+ \int_0^t D^2 U(s, X(s))\, D_h X(s)\, dW(s). \end{aligned}$$

All terms are meaningful (for instance the tensor valued coefficient $D^2 U(s, X(s))$ is bounded continuous), $\nabla U(t, X(t))$ has norm less than $1/2$ (hence the term $\nabla U(t, X(t))\, D_h X(t)$ contracts) and one can prove that this equation has a solution $D_h X(s)$. Along these lines one can build a rigorous proof of differentiability. We do not discuss the other properties.

Remark 3 A main open problem is the case when b is random:$b = b(\omega, t, x)$. In this case, strong uniqueness statements of the previous form are unknown (when b is not regular).

1.2 The PDE case

We have seen that noise improves the theory of ODEs. Is it the same for PDEs? We have several more possibilities, several dichotomies:

$$\text{equations:} \quad \begin{array}{l} \nearrow \ \text{linear} \\ \searrow \ \text{non linear} \end{array}$$

$$\text{problems:} \quad \begin{array}{l} \nearrow \ \text{uniqueness (weak solutions)} \\ \searrow \ \text{blow-up (regular solutions)} \end{array}$$

$$\text{noise:} \quad \begin{array}{l} \nearrow \ \text{additive (like for ODEs)} \\ \searrow \ \text{bilinear multiplicative.} \end{array}$$

Let us deal with two of the simplest but not trivial combinations: *linear* transport equations, both the problem of *uniqueness* of weak L^∞ solutions and of *no blow-up* of C^1-solutions, the improvements of the deterministic theory produced by a *bilinear multiplicative* noise.

The linear deterministic transport equation is the first order PDE in $\mathbb{R}^d$

$$\frac{\partial u}{\partial t} + b \cdot \nabla u = 0, \qquad u|_{t=0} = u_0$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is given and we look for a solution $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$.

Definition 4 *Assume $b, \operatorname{div} b \in L^1_{loc} = L^1_{loc}([0,T] \times \mathbb{R}^d)$, $u_0 \in L^\infty\left(\mathbb{R}^d\right)$. We say that u is a weak L^∞-solution if:*

i) $u \in L^\infty\left([0,T] \times \mathbb{R}^d\right)$

ii) for all $\theta \in C_0^\infty\left(\mathbb{R}^d\right)$ one has

$$\int_{\mathbb{R}^d} u(t,x)\,\theta(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\,\theta(x)\,dx + \int_0^t \int_{\mathbb{R}^d} u(s,x) \operatorname{div}\left(b(s,x)\,\theta(x)\right) dx ds$$

Existence of weak L^∞-solutions is a general fact, obtained by weak-star compactness methods. When $b \in L^\infty\left(0,T; Lip_b\left(\mathbb{R}^d;\mathbb{R}^d\right)\right)$, uniqueness can be proved, and also existence of smoother solutions when u_0 is smoother. Moreover, one has the transport relation

$$u(t, \phi(t,x)) = u_0(x)$$

where $\phi(t,x)$ is the deterministic flow associated to the equation of characteristics

$$\frac{d}{dt}\phi(t,x) = b(\phi(t,x)), \quad \phi(0,x) = x.$$

When b is less than Lipschitz continuous, there are counterexamples. For instance, for

$$d = 1, \quad b(x) = 2sign(x)\sqrt{|x|}$$

the PDE has infinitely many solutions from any initial condition u_0. These solutions coincide for $|x| > t^2$, where the flow is uniquely defined, but they can be prolonged almost arbitrarily for $|x| < t^2$, for instance setting

$$u(t,x) = C \text{ for } |x| < t^2$$

with arbitrary C. Remarkable is the result of [5] which states that the solution is unique when (we do not stress the generality of the behavior at infinity)

$$\nabla b \in L^1_{loc}\left([0,T] \times \mathbb{R}^d; \mathbb{R}^d\right), \tag{2}$$

$$\operatorname{div} b \in L^1\left(0,T; L^\infty\left(\mathbb{R}^d, \mathbb{R}^d\right)\right). \tag{3}$$

There are generalizations of this result (for instance [1]), but not so far from it. In these cases the flow exists and is unique but only in a proper generalized sense. The assumption (3) is the quantitative one used to prove the estimate (for simplicity we omit the cut-of needed to localize)

$$\int_{\mathbb{R}^d} u^2(t,x)\,dx = \int_{\mathbb{R}^d} u_0^2(x)\,dx + \int_0^t ds \int_{\mathbb{R}^d} u^2(s,x)\operatorname{div} b(s,x)\,dx$$

$$\leq \int_{\mathbb{R}^d} u_0^2(x)\,dx + \int_0^t \|\operatorname{div} b(s,\cdot)\|_\infty\,ds \int_{\mathbb{R}^d} u^2(s,x)\,dx$$

which implies, by Gronwall lemma, $\int_{\mathbb{R}^d} u^2(t,x)\,dx = 0$ when $u_0 = 0$ (this implies uniqueness, since the equation is linear). The assumption (2) apparently has no role but it is essential to perform these computations rigorously. One has to prove that a weak L^∞-solution u satisfies the previous identity. In order to apply differential calculus to u, one can mollify u but then a remainder, a commutator, appears in the equation. The convergence to zero of this commutator (established by the so called *commutator lemma* of [5]) requires assumption (2). We have recalled these facts since they are a main motiv below.

The problem of no blow-up of C^1 or $W^{1,p}$ solution is open for the deterministic equation, under essentially weaker conditions than Lipschitz continuity of b. The equation satisfied by first derivatives $v_k = \frac{\partial u}{\partial x_k}$ involves derivatives of b as a potential term

$$\frac{\partial v_k}{\partial t} + b \cdot \nabla v_k + \sum_i \frac{\partial b}{\partial x_i} v_i = 0, \qquad v_k|_{t=0} = \frac{\partial u_0}{\partial x_k}$$

and L^{∞} bounds on $\frac{\partial b}{\partial x_i}$ seem necessary to control v_k. Again there are simple counterexamples: in the case

$$d = 1, \quad b(x) = -2sign(x)\sqrt{|x|},$$

the equation of characteristics has coalescing trajectories (the solutions from $\pm x_0$ meet at $x = 0$ at time $\sqrt{|x_0|}$) and thus, if we start with a smooth initial condition u_0 such that at some point x_0 satisfies $u_0(x_0) \neq u_0(-x_0)$, then at time $t_0 = \sqrt{|x_0|}$ the solution is discontinuous (unless u_0 is special, the discontinuity appears immediately, for $t > 0$).

Consider the following stochastic version of the linear transport equation:

$$\frac{\partial u}{\partial t} + b \cdot \nabla u + \sigma \nabla u \circ \frac{dW}{dt} = 0, \qquad u|_{t=0} = u_0.$$

The noise W is a d-dimensional Brownian motion, $\sigma \in \mathbb{R}$, the operation $\nabla u \circ \frac{dW}{dt}$ has simultaneously two features: it is a scalar product between the vectors ∇u and $\frac{dW}{dt}$, and has to be interpreted in the Stratonovich sense. The noise has a transport structure as the deterministic part of the equation. It is like to add the fast oscillating term $\sigma \frac{dW}{dt}$ to the drift b:

$$b(x) \longrightarrow b(x) + \sigma \frac{dW}{dt}(t).$$

Concerning Stratonovich calculus and its relation with Itô calculus, see [15]. We recall the so called Wong-Zakai principle (proved as a rigorous theorem in several cases): when one takes a differential equations with a smooth approximation of Brownian motion, and then takes the limit towards true Brownian motion, the correct limit equation involves Stratonovich integrals. Thus equations with Stratonovich integrals are more physically based.

Definition 5 *Assume $b, \operatorname{div} b \in L^1_{loc}$, $u_0 \in L^{\infty}(\mathbb{R}^d)$. We say that a stochastic process u is a weak L^{∞}-solution of the SPDE if:*

i) $u \in L^{\infty}(\Omega \times [0,T] \times \mathbb{R}^d)$

ii) for all $\theta \in C_0^{\infty}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} u(t,x)\theta(x)\,dx$ is a continuous adapted semimartingale

iii) for all $\theta \in C_0^{\infty}(\mathbb{R}^d)$, one has

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,x)\theta(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\theta(x)\,dx + \int_0^t \int_{\mathbb{R}^d} u(s,x) \operatorname{div}(b(s,x)\theta(x))\,dxds \\ + \sigma \int_0^t \left(\int_{\mathbb{R}^d} u(s,x) \nabla\theta(x)\,dx \right) \circ dW(s).\end{aligned}$$

The following theorem is due to [8].

Theorem 6 *If $\sigma \neq 0$ and*

$$b \in L^{\infty}\left(0,T;C_b^{\alpha}\left(\mathbb{R}^d;\mathbb{R}^d\right)\right), \quad \operatorname{div} b \in L^p([0,T]\times\mathbb{R}^d), \tag{4}$$

for some $\alpha \in (0,1)$ and $p > d\wedge 2$, then there exists a unique weak L^{∞}-solution of the SPDE. If $\alpha \in (1/2,1)$ then we have uniqueness only assuming $\operatorname{div} b \in L^1_{loc}$. Moreover, it holds

$$u\left(t,\phi\left(t,x\right)\right) = u_0\left(x\right)$$

where $\phi\left(t,x\right)$ is the stochastic flow of diffeomorphisms associated to the equation

$$d\phi\left(t,x\right) = b\left(t,\phi\left(t,x\right)\right)dt + \sigma dW\left(t\right), \quad \phi\left(0,x\right) = x$$

given by Theorem 2.

Thus we see that a suitable noise improves the theory of linear transport equation from the view-point of uniqueness of weak solutions. One of the aims of this paper is to prove a variant of this theorem, under different assumptions on b. It requires a new form of commutator lemma with respect to those proved in [5] or [8].

Let us come to the blow-up problem. The following result can be deduced from [8, Appendix A] in which we have considered BV_{loc}-solutions for the transport equation. In Section 2 we will give a direct proof of the existence part which is of independent interest.

Theorem 7 *If $\sigma \neq 0$,*

$$b \in L^{\infty}(0,T;C_b^{\alpha}(\mathbb{R}^d;\mathbb{R}^d)),$$

for some $\alpha \in (0,1)$ and $u_0 \in C_b^1\left(\mathbb{R}^d\right)$, then there exists a unique classical C^1-solution for the transport equation with probability one. It is given by

$$u\left(t,x\right) = u_0\left(\phi_t^{-1}\left(x\right)\right) \tag{5}$$

where ϕ_t^{-1} is the inverse of the stochastic flow $\phi_t = \phi\left(t,\cdot\right)$.

The main claim of this theorem is the regularity of the solution for positive times, which is new with respect to the deterministic case. The uniqueness claim is known, as a particular case of a result in BV_{loc}, see Appendix 1 of [8].

Notice that, for solutions with such degree of regularity (BV_{loc} or C^1), no assumption on $\operatorname{div} b$ is required; $\operatorname{div} b$ does not even appear in the definition of solution (see below). On the contrary, to reach uniqueness in the much wider class of weak L^{∞}-solutions, in [8] we had to impose the additional condition (4) on $\operatorname{div} b$, for some $p > d \wedge 2$ ($\operatorname{div} b$ also appears in the definition of weak L^{∞}-solution); this happens also in the deterministic theory.

1.3 Some other works on regularization by noise

The following list does not aim to be exhaustive, see for instance [7] for other results and references:

- the uniqueness for linear transport equations can be extended to other weak assumptions on the drift, [2], [16]; also no blow-up holds for L^p drift see [6] and [18];
- similar results hold for linear continuity equations, [9], [17]:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(b\rho) = 0, \qquad \rho|_{t=0} = \rho_0 :$$

  a noise of the form $\nabla \rho \circ \frac{dW}{dt}$ prevents mass concentration;

- analog results hold for the vector valued linear equations

$$\frac{\partial M}{\partial t} + \mathrm{curl}\,(b \times M) = 0$$

  similar to the vorticity formulation of 3D Euler equations or magneto-hydrodynamics, where the singularities in the deterministic case are not shocks but infinite values of M; a noise of the form $\mathrm{curl}\,(e \times M) \circ \frac{dW}{dt}$ prevents blow-up [12];

- improved Strichartz estimates for a special Schrödinger model with noise

$$i\partial_t u + \Delta u \circ \frac{dW}{dt} = 0$$

  have been proved, which are stronger than the corresponding ones for $i\partial_t u + \Delta u = 0$ and allow to prevent blow-up in a non-linear case when blow-up is possible without noise, see [3];

- nonlinear transport type equations of two forms have been investigated: 2D Euler equations and 1D Vlasov-Poisson equations; in these cases non-collapse of measure valued solutions concentrated in a finite number of points has been proved, [11], [4].

We conclude the introduction with some notations.

1.4 Notations

Usually we denote by $D_i f$ the derivative in the i-th coordinate direction and with $(e_i)_{i=1,\dots,d}$ the canonical basis of $\mathbb{R}^d$ so that $D_i f = e_i \cdot Df$. For partial derivatives of any order $n \geq 1$ we use the notation $D^n_{i_1,\dots,i_n}$. If $\eta : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-diffeomorphism we will denote by $J\eta(x) = \det[D\eta(x)]$ its Jacobian determinant. For a given function f depending on $t \in [0,T]$ and $x \in \mathbb{R}^d$, we will also adopt the notation $f_t(x) = f(t,x)$.

Let $T > 0$ be fixed. For $\alpha \in (0,1)$ define the space $L^\infty\left(0,T; C_b^\alpha(\mathbb{R}^d)\right)$ as the set of all bounded Borel functions $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ for which

$$[f]_{\alpha,T} = \sup_{t\in[0,T]} \sup_{x\neq y\in\mathbb{R}^d} \frac{|f(t,x)-f(t,y)|}{|x-y|^\alpha} < \infty$$

($|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ for every d, if no confusion may arise). This is a Banach space with respect to the usual norm $\|f\|_{\alpha,T} = \|f\|_0 + [f]_{\alpha,T}$ where $\|f\|_0 = \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |f(t,x)|$. Similarly, when $\alpha = 1$ we define $L^\infty\left(0,T; Lip_b(\mathbb{R}^d)\right)$.

We write $L^\infty\left(0,T; C_b^\alpha(\mathbb{R}^d;\mathbb{R}^d)\right)$ for the space of all vector fields $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ having all components in $L^\infty\left(0,T; C_b^\alpha(\mathbb{R}^d)\right)$.

Moreover, for $n \geq 1$, $f \in L^\infty\left(0,T; C_b^{n+\alpha}(\mathbb{R}^d)\right)$ if all spatial partial derivatives $D^k_{i_1,\ldots,i_k} f \in L^\infty\left(0,T; C_b^\alpha(\mathbb{R}^d)\right)$, for all orders $k = 0,1,\ldots,n$. Define the corresponding norm as

$$\|f\|_{n+\alpha,T} = \|f\|_0 + \sum_{k=1}^n \|D^k f\|_0 + [D^n f]_{\alpha,T},$$

where we extend the previous notations $\|\cdot\|_0$ and $[\cdot]_{\alpha,T}$ to tensors. The definition of the space $L^\infty\left(0,T; C_b^{n+\alpha}(\mathbb{R}^d;\mathbb{R}^d)\right)$ is similar. The spaces $C_b^{n+\alpha}(\mathbb{R}^d)$ and $C_b^{n+\alpha}(\mathbb{R}^d;\mathbb{R}^d)$ are defined as before but only involve functions $f : \mathbb{R}^d \to \mathbb{R}^d$ which do not depend on time. Moreover, we say that $f : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $C^{n,\alpha}$, $n \in \mathbb{N}$, $\alpha \in (0,1)$, if f is continuous on $\mathbb{R}^d$, n-times differentiable with all continuous derivatives and the derivatives of order n are locally α-Hölder continuous. Finally, $C_0^0(\mathbb{R}^d)$ denotes the space of all real continuous functions defined on $\mathbb{R}^d$, having compact support and by $C_0^\infty(\mathbb{R}^d)$ its subspace consisting of infinitely differentiable functions.

For any $r > 0$ we denote by $B(r)$ the Euclidean ball centered in 0 of radius r and by $C_r^\infty(\mathbb{R}^d)$ the space of smooth functions with compact support in $B(r)$; moreover, $\|\cdot\|_{L_r^p}$ and $\|\cdot\|_{W_r^{1,p}}$ stand for, respectively, the L^p-norm and the $W^{1,p}$-norm on $B(r)$, $p \in [1,\infty]$. We let also $[f]_{C_r^\theta} = \sup_{x\neq y\in B(r)} |f(x)-f(y)|/|x-y|^\theta$.

We will often use the standard mollifiers. Let $\vartheta : \mathbb{R}^d \to \mathbb{R}$ be a smooth test function such that $0 \leq \vartheta(x) \leq 1$, $x \in \mathbb{R}^d$, $\vartheta(x) = \vartheta(-x)$, $\int_{\mathbb{R}^d} \vartheta(x)dx = 1$, $\operatorname{supp}(\vartheta) \subset B(2)$, $\vartheta(x) = 1$ when $x \in B(1)$. For any $\varepsilon > 0$, let $\vartheta_\varepsilon(x) = \varepsilon^{-d}\vartheta(x/\varepsilon)$ and for any distribution $g : \mathbb{R}^d \to \mathbb{R}^n$ we define the mollified approximation g^ε as

$$g^\varepsilon(x) = \vartheta_\varepsilon * g(x) = g(\vartheta_\varepsilon(x-\cdot)), \quad x \in \mathbb{R}^d. \tag{6}$$

If g depends also on time t, we consider $g^\varepsilon(t,x) = (\vartheta_\varepsilon * g(t,\cdot))(x)$, $t \in [0,T]$, $x \in \mathbb{R}^d$.

Recall that, for any smooth bounded domain $\mathcal{D}$ of $\mathbb{R}^d$, we have: $f \in W^{\theta,p}(\mathcal{D})$, $\theta \in (0,1)$, $p \geq 1$, if and only if $f \in L^p(\mathcal{D})$ and

$$[f]^p_{W^{\theta,p}} = \iint_{\mathcal{D}\times\mathcal{D}} \frac{|f(x)-f(y)|^p}{|x-y|^{\theta p+d}} dxdy < \infty.$$

We have $W^{1,p}(\mathcal{D}) \subset W^{\theta,p}(\mathcal{D})$, $\theta \in (0,1)$.

In the sequel we will assume a stochastic basis with a d-dimensional Brownian motion $(\Omega, (\mathcal{F}_t), \mathcal{F}, P, (W_t))$ to be given. We denote by $\mathcal{F}_{s,t}$ the completed σ-algebra generated by $W_u - W_r$, $s \le r \le u \le t$, for each $0 \le s < t$.

Let us finally recall our basic assumption on the drift vector field.

Hypothesis 1 *There exists $\alpha \in (0,1)$ such that $b \in L^\infty\left(0,T; C_b^\alpha(\mathbb{R}^d;\mathbb{R}^d)\right)$.*

2 No blow-up in C^1

This section is devoted to prove Theorem 7. Since the solution claimed by this theorem is regular, we do not need to integrate over test functions in the term $b \cdot \nabla u$ and thus we do not need to require $\operatorname{div} b \in L^1_{loc}$. For this reason, we modify the definition of solution.

Definition 8 *Assume $b \in L^1_{loc}$, $u_0 \in C_b^1\left(\mathbb{R}^d\right)$. We say that a stochastic process $u \in L^\infty(\Omega \times [0,T] \times \mathbb{R}^d)$ is a classical C^1-solution of the stochastic transport equation if:*

i) $u(\omega,t,\cdot) \in C^1(\mathbb{R}^d)$ for a.e. $(\omega,t) \in \Omega \times [0,T]$;

ii) for all $\theta \in C_0^\infty\left(\mathbb{R}^d\right)$, $\int_{\mathbb{R}^d} u(t,x)\theta(x)\,dx$ is a continuous adapted semi-martingale;

iii) for all $\theta \in C_0^\infty\left(\mathbb{R}^d\right)$, one has

$$\int_{\mathbb{R}^d} u(t,x)\,\theta(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\,\theta(x)\,dx - \int_0^t \int_{\mathbb{R}^d} b(s,x) \cdot \nabla u(s,x)\,\theta(x)\,dxds + \sigma \int_0^t \left(\int_{\mathbb{R}^d} u(s,x)\,\nabla\theta(x)\,dx \right) \circ dW(s).$$

If u is a classical C^1-solution and $\operatorname{div} b \in L^1_{loc}\left([0,T] \times \mathbb{R}^d\right)$, then u is also a weak L^∞-solution. Conversely, if u is a weak L^∞-solution, $u_0 \in C_b^1\left(\mathbb{R}^d\right)$ and (i) is satisfied then u is a classical C^1-solution.

Before giving the proof we mention the following useful result proved in [8, Theorem 5]:

Theorem 9 *Assume that Hypothesis 1 holds true for some $\alpha \in (0,1)$. Then we have the following facts:*

(i) (pathwise uniqueness) For every $s \in [0,T]$, $x \in \mathbb{R}^d$, the stochastic equation (1) has a unique continuous adapted solution $X^{s,x} = \left(X_t^{s,x}(\omega), t \in [s,T], \omega \in \Omega\right)$.

(ii) (differentiable flow) There exists a stochastic flow $\phi_{s,t}$ of diffeomorphisms for equation (1). The flow is also of class $C^{1+\alpha'}$ for any $\alpha' < \alpha$.

(iii) (stability) Let $(b^n) \subset L^\infty\left(0,T;C_b^\alpha(\mathbb{R}^d;\mathbb{R}^d)\right)$ be a sequence of vector fields and ϕ^n be the corresponding stochastic flows. If $b^n \to b$ in $L^\infty(0,T;C_b^{\alpha'}(\mathbb{R}^d;\mathbb{R}^d))$ for some $\alpha' > 0$, then, for any $p \geq 1$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{0\le s\le T} E[\sup_{r\in[s,T]} |\phi_{s,r}^n(x) - \phi_{s,r}(x)|^p] = 0 \tag{7}$$

$$\sup_{n\in\mathbb{N}} \sup_{x\in\mathbb{R}^d} \sup_{0\le s\le T} E[\sup_{u\in[s,T]} \|D\phi_{s,u}^n(x)\|^p] < \infty, \tag{8}$$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{0\le s\le T} E[\sup_{r\in[s,T]} \|D\phi_{s,r}^n(x) - D\phi_{s,r}(x)\|^p] = 0. \tag{9}$$

Remark 10 We point out that the previous assertions (7), (8) and (9) also holds when $\phi_{s,r}^n(x)$ and $\phi_{s,r}(x)$ are replaced respectively by $(\phi_{s,r}^n)^{-1}(x)$ and $(\phi_{s,r})^{-1}(x)$.

To see this note that for a fixed $t > 0$, $Z_s = (\phi_{s,t})^{-1}(x)$, $s \in [0,t]$, is measurable with respect to $\mathcal{F}_{s,t}$ (the completed σ-algebra generated by $W_u - W_r$, $s \le r \le u \le t$, for each $0 \le s < t$) and solves

$$Z_s = x - \int_s^t b(r, Z_r)dr - \sigma[W_t - W_s]. \tag{10}$$

This is a simple backward stochastic differential equations, of the same form as the original one (only the drift has opposite sign). Note that for regular functions $f \in C_b^2(\mathbb{R}^d)$, Itô's formula becomes

$$f(Z_s) = f(x) - \int_s^t \nabla f(Z_r)\cdot b(r,Z_r)dr - \int_s^t \nabla f(Z_r)\cdot dW_r - \frac{\sigma^2}{2}\int_s^t \triangle f(Z_r)dr$$

where $\int_s^t \nabla f(Z_r)\cdot dW_r$ is the so called backward Itô integral (is a limit in probability of elementary integrals like $\sum_k \nabla f(Z_{s_k})\cdot(W_{s_k} - W_{s_{k-1}})$ in which we consider the partition $s_0 = 0 < \ldots < s_N = t$). Since this stochastic integral enjoys usual properties of the classical Itô integral, one can repeat all the arguments needed to prove (7), (8) and (9) even for solutions Z to (10).

Proof. (**Theorem 7**) Under the assumptions of the theorem, it has been proved in Appendix 1 of [8] that unqueness holds in BV_{loc}. Hence it holds in C^1. For this result, no assumption on $\operatorname{div} b$ is required.

We show now that (5) is a classical C^1-solution. It is easy to check (i) in Definition 8. Moreover, if $\theta \in C_0^\infty(\mathbb{R}^d)$, by changing variable we have:

$$\int_{\mathbb{R}^d} u(t,x)\theta(x)dx = \int_{\mathbb{R}^d} u_0(y)\theta(\phi_t(y))J\phi_t(y)dy,$$

where $J\phi_t(y) = \det[D\phi_t(y)]$, and so also property (ii) follows. To prove property (iii) consider the flow ϕ_t^ε for the regularized vector field b^ε (see (6)) and let $J\phi_t^\varepsilon(y)$ be its Jacobian determinant. Note that $u_0 \circ (\phi_t^\varepsilon)^{-1} \to u_0 \circ \phi_t^{-1}$ weakly in $L^\infty(\mathbb{R}^d)$, uniformly in $t \in [0,T]$ and P-a.s., indeed for $\theta \in C_0^\infty(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} (u_0 \circ (\phi_t^\varepsilon)^{-1})(y)\theta(y)dy = \int_{\mathbb{R}^d} u_0(y)\theta(\phi_t^\varepsilon(y))J\phi_t^\varepsilon(y)dy$$

$$\to \int_{\mathbb{R}^d} u_0(y)\theta(\phi_t(y))J\phi_t(y)dy,$$

as $\epsilon \to 0$, using the properties of the stochastic flow stated in Theorem 9. By density we can extend this convergence to any $\theta \in L^1(\mathbb{R}^d)$. Moreover since b^ε is smooth, it is easy to prove that

$$dJ_t^\varepsilon(y) = \operatorname{div} b_t^\varepsilon(\phi_t^\varepsilon(y))J\phi_t^\varepsilon(y)dt$$

and by the Itô formula we find

$$\begin{aligned}
&\int_{\mathbb{R}^d} u_0(y)\theta(\phi_t^\varepsilon(y))J_t^\varepsilon(y)dy = \\
&\qquad \int_{\mathbb{R}^d} u_0(y)\theta(y)dy + \int_0^t ds \int_{\mathbb{R}^d} u_0(y)L^{b^\varepsilon}\theta(\phi_s^\varepsilon(y))J\phi_s^\varepsilon(y)dy \\
&\qquad + \int_0^t ds \int_{\mathbb{R}^d} u_0(y)\theta(\phi_s^\varepsilon(y))\operatorname{div} b_s^\varepsilon(\phi_s^\varepsilon(y))J\phi_s^\varepsilon(y)dy \\
&\qquad + \sigma \int_0^t dW_s \cdot \int_{\mathbb{R}^d} u_0(y)\nabla\theta(\phi_s^\varepsilon(y))J\phi_s^\varepsilon(y)dy,
\end{aligned} \tag{11}$$

where

$$L^{b^\varepsilon}\theta(y) = \frac{1}{2}\sigma^2\Delta\theta(y) + b_s^\epsilon(y)\cdot\nabla\theta(y).$$

Note that, integrating by parts,

$$\begin{aligned}
&\int_0^t ds \int_{\mathbb{R}^d} u_0(y)\theta(\phi_s^\varepsilon(y))\operatorname{div} b_s^\varepsilon(\phi_s^\varepsilon(y))J\phi_s^\varepsilon(y)dy \\
&= \int_0^t ds \int_{\mathbb{R}^d} u_0((\phi_s^\varepsilon)^{-1}(x))\theta(x)\operatorname{div} b_s^\varepsilon(x)dx \\
&= -\int_0^t ds \int_{\mathbb{R}^d} u_0((\phi_s^\varepsilon)^{-1}(x))\nabla\theta(x)\cdot b_s^\varepsilon(x)dx \\
&\quad -\int_0^t ds \int_{\mathbb{R}^d} \nabla u_0((\phi_s^\varepsilon)^{-1}(x))D(\phi_s^\varepsilon)^{-1}(x)\cdot b_s^\varepsilon(x)\,\theta(x)dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^d} u_0(y)\theta(\phi_t^\varepsilon(y))J_t^\varepsilon(y)dy = \\
&\qquad \int_{\mathbb{R}^d} u_0(y)\theta(y)dy + \frac{1}{2}\sigma^2\int_0^t ds\int_{\mathbb{R}^d} u_0((\phi_s^\varepsilon)^{-1}(x))\triangle\theta(x)dx \\
&\qquad - \int_0^t ds \int_{\mathbb{R}^d} \nabla u_0((\phi_s^\varepsilon)^{-1}(x))D(\phi_s^\varepsilon)^{-1}(x)\cdot b_s^\varepsilon(x)\,\theta(x)dx \\
&\qquad + \sigma\int_0^t dW_s \cdot \int_{\mathbb{R}^d} u_0(y)\nabla\theta(\phi_s^\varepsilon(y))J\phi_s^\varepsilon(y)dy.
\end{aligned}$$

By changing variable $y = (\phi_s^\varepsilon)^{-1}(x)$ of the second and third integral in the right-hand side, there are no problems to pass to the limit as $\epsilon \to 0$, $\mathbb{P}$-a.s., using (iii) in Theorem 9 and Remark 10 (precisely, one can pass to the limit along a suitable sequence $(\epsilon_n) \subset (0,1)$ converging to 0). To this purpose we only note that for the stochastic integral we have

$$\int_0^t dW_s \cdot \int_{\mathbb{R}^d} u_0(y) \nabla\theta(\phi_s^\varepsilon(y)) J\phi_s^\varepsilon(y) dy \;\to\; \int_0^t dW_s \cdot \int_{\mathbb{R}^d} u_0(y) \nabla\theta(\phi_s(y)) J\phi_s(y) dy$$

uniformly on $[0,T]$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Finally we get

$$\begin{aligned}\int_{\mathbb{R}^d} u_0((\phi_t)^{-1}(x))\theta(x)dx = \int_{\mathbb{R}^d} u_0(y)\theta(y)dy + \frac{\sigma^2}{2}\int_0^t ds \int_{\mathbb{R}^d} u_0((\phi_s)^{-1}(x))\triangle\theta(x)dx \\ - \int_0^t ds \int_{\mathbb{R}^d} \nabla u_0((\phi_s)^{-1}(x)) D(\phi_s)^{-1}(x) \cdot b_s(x)\,\theta(x)dx \\ + \sigma \int_0^t dW_s \cdot \int_{\mathbb{R}^d} u_0((\phi_s)^{-1}(x))\nabla\theta(x)dx.\end{aligned}$$

By passing from Itô to Stratonovich integral this is exactly the formula we wanted to prove. The proof is complete. ■

Remark 11 One can show that the boundedness assumption on b is not important to prove the previous Theorem 7. Indeed at least when b is independent on t, one can prove the result with b possibly unbounded, only assuming that its component b_i are "locally uniformly α-Hölder continuous", i.e.,

$$[b_i]_{\alpha,1} := \sup_{x \neq y \in \mathbb{R}^d} \frac{|b_i(x) - b_i(y)|}{(|x-y|^\alpha \vee |x-y|)} < +\infty, \quad i = 1, \ldots, d, \tag{12}$$

where $a \vee b = \max(a,b)$, for $a, b \in \mathbb{R}$. Under (12) one can still construct a stochastic differentiable flow $\phi_t(x)$ (see Theorem 7 in [10]) which satisfies properties (8) and (9) (see also Remark 10) and this allows to perform the same proof of Theorem 7.

3 A stability property

The following result shows a *stability property* for the solutions of the SPDE; such property involves the *weak∗* topology (or the $\sigma(L^\infty(\mathbb{R}^d), L^1(\mathbb{R}^d))$-topology).

Proposition 12 *Assume that Hypothesis 1 holds true for some $\alpha \in (0,1)$. Moreover, denote by $\phi_t = \phi_{0,t}$ the stochastic flow for equation (1). Then, for any sequence $(v^n) \subset L^\infty(\mathbb{R}^d)$, we have:*

$$v_n \to v \in L^\infty(\mathbb{R}^d) \text{ in weak}* \text{ topology} \implies v_n(\phi_t^{-1}(\cdot)) \to v(\phi_t^{-1}(\cdot))$$

in weak∗ topology,

uniformly in $t \in [0,T]$, $P-a.s.$

Proof. We prove that, P-a.s., for any $f \in L^1(\mathbb{R}^d)$ we have

$$a_n = \sup_{t\in[0,T]} \Big| \int_{\mathbb{R}^d} [v_n(\phi_t^{-1}(y)) - v(\phi_t^{-1}(y))]\, f(y)dy \Big| \to 0, \tag{13}$$

as $n \to \infty$.

Recall that there exists a positive constant M such that $\|v_n\|_0 \le M$, $n \ge 1$, and $\|v\|_0 \le M$ and, moreover, by the separability of $L^1(\mathbb{R}^d)$ there exists a countable dense set $D \subset C_0^\infty(\mathbb{R}^d)$.

It is enough to check (13) when $f \in D$ (with the event of probability one, possibly depending on f). Indeed, if $f \in L^1(\mathbb{R}^d)$, we can consider a sequence $(f_N) \subset D$ which converges to f in $L^1(\mathbb{R}^d)$ and find, $P-$a.s.,

$$a_n \le 2M \int_{\mathbb{R}^d} |f(y) - f_N(y)|dy + \sup_{t\in[0,T]} \Big| \int_{\mathbb{R}^d} [v_n(\phi_t^{-1}(y)) - v(\phi_t^{-1}(y))]\, f_N(y)dy \Big|;$$

by the previous inequality the assertion follows easily.

To prove (13) for a fixed $f \in D$ we first note that, by changing variable ($J\phi_t(x)$ denotes the Jacobian determinant of ϕ_t at x)

$$\int_{\mathbb{R}^d} [v_n(\phi_t^{-1}(y)) - v(\phi_t^{-1}(y))]\, f(y)dy = \int_K [v(x) - v_n(x)]\, f(\phi_t(x)) J\phi_t(x)dx, \tag{14}$$

where we have defined the compact set $K = \pi_2(\{(t,x) \in [0,T]\times\mathbb{R}^d \ : \ \phi_t^{-1}(x) \in \mathrm{supp}(f)\})$, with $\pi_2(s,x) = x$, $s \in [0,T]$, $x \in \mathbb{R}^d$.

Using that, P-a.s., the map: $(t,x) \mapsto f(\phi_t(x))J\phi_t(x)$ is continuous on $[0,T]\times\mathbb{R}^d$, we see from (14) that the map: $t \mapsto \int_{\mathbb{R}^d} [v_n(\phi_t^{-1}(y)) - v(\phi_t^{-1}(y))]\, f(y)dy$ is continuous on $[0,T]$ and so, P-a.s.,

$$a_n = \sup_{t\in[0,T]\cap\mathbb{Q}} \Big| \int_{\mathbb{R}^d} [v_n(\phi_t^{-1}(y)) - v(\phi_t^{-1}(y))]\, f(y)dy \Big|. \tag{15}$$

By (14) we also deduce that, P-a.s.,

$$\Big| \int_{\mathbb{R}^d} [v_n(\phi_t^{-1}(y)) - v(\phi_t^{-1}(y))]\, f(y)dy \Big| \to 0, \quad t \in [0,T]\cap\mathbb{Q}. \tag{16}$$

We finish the proof arguing by contradiction. We consider an event Ω_0 with $P(\Omega_0) = 1$ such that (15), (16) holds for any $\omega \in \Omega_0$ and also $(t,x) \mapsto f(\phi(t,\omega)(x))J\phi(t,\omega)(x)$ is continuous on $[0,T]\times\mathbb{R}^d$ for any $\omega \in \Omega_0$.

If (13) does not hold for some $\omega_0 \in \Omega_0$, then there exists $\varepsilon > 0$ and $(t_n) \subset [0,T]\cap\mathbb{Q}$ such that

$$\Big| \int_{\mathbb{R}^d} [v_n(\phi_{t_n}^{-1}(y)) - v(\phi_{t_n}^{-1}(y))]\, f(y)dy \Big| > \varepsilon$$

(we do not indicate dependence on ω_0 to simplify notation; in the sequel we always argue at ω_0 fixed). Possibly passing to a subsequence, we may assume that $t_n \to \hat{t} \in [0,T]$.

By changing variable we have, for any $n \geq 1$,

$$\varepsilon < \Big| \int_K [v(x) - v_n(x)] \, f(\phi_{t_n}(x)) J\phi_{t_n}(x) dx \Big| \leq (1) + (2),$$

$$(1) = \Big| \int_K [v(x) - v_n(x)] \, [f(\phi_{t_n}(x)) J\phi_{t_n}(x) - f(\phi_{\hat{t}}(x)) J\phi_{\hat{t}}(x)] dx \Big|,$$

$$(2) = \Big| \int_K [v(x) - v_n(x)] \, f(\phi_{\hat{t}}(x)) J\phi_{\hat{t}}(x) dx \Big|.$$

Now

$$(1) \leq 2M \int_K |f(\phi_{t_n}(x)) J\phi_{t_n}(x) - f(\phi_{\hat{t}}(x)) J\phi_{\hat{t}}(x)| dx,$$

which tends to 0, as $n \to \infty$, $P-$a.s., by the dominated convergence theorem (indeed at ω_0 fixed, $(t, x) \mapsto f(\phi(t, \omega_0)(x)) J\phi(t, \omega_0)(x)$ is continuous on $[0, T] \times \mathbb{R}^d$).

Let us consider (2). By uniform continuity of $f(\phi_t(x)) J\phi_t(x)$ on $[0, T] \times K$ we may choose $q \in [0, T] \cap \mathbb{Q}$ such that

$$|f(\phi_{\hat{t}}(x)) J\phi_{\hat{t}}(x) - f(\phi_q(x)) J\phi_q(x)| < \frac{\epsilon}{4M \, \lambda(K)},$$

for any $x \in K$ (here $\lambda(K)$ is the Lebesgue measure of K). Now, for any $n \geq 1$,

$$\begin{aligned}
(2) &\leq \Big| \int_K [v(x) - v_n(x)] \, [f(\phi_{\hat{t}}(x)) J\phi_{\hat{t}}(x) - f(\phi_q(x)) J f(\phi_q(x))] dx \Big| \\
&+ \Big| \int_K [v(x) - v_n(x)] \, f(\phi_q(x)) J\phi_q(x) dx \Big| \\
&\leq \epsilon/2 + \Big| \int_K [v(x) - v_n(x)] \, f(\phi_q(x)) J\phi_q(x) dx \Big|.
\end{aligned}$$

Since $x \mapsto f(\phi_q(x)) J\phi_q(x)$ is integrable on $\mathbb{R}^d$, we find that the last term tends tends to 0, as $n \to \infty$.

We have found a contradiction. The proof is complete. ■

4 New uniqueness results

The aim of this section is to prove some new uniqueness results for L^∞ weak solutions of the SPDE obtained extending the key estimates in fractional Sobolev spaces.

Unlike Theorem 7 we will assume more conditions on b. On the other hand we will allow $u_0 \in L^\infty(\mathbb{R}^d)$ and prove stronger uniqueness results in the larger class of weak solutions. Recall that the uniqueness statement, in a class of so regular solutions, of Theorem 7 is rather obvious and does not require special effort and assumptions on the drift. On the contrary, the uniqueness claims in a class of weak solutions of Theorems 13 and 14 below are quite delicate and require suitable conditions on the drift.

The first result is the following:

Theorem 13 *Let $d \geq 2$ and $u_0 \in L^\infty(\mathbb{R}^d)$. Assume Hypothesis 1 and also that*

$$\operatorname{div} b \in L^q(0,T;L^p(\mathbb{R}^d))$$

for some $q > 2 \geq p > \frac{2d}{d+2\alpha}$. Then there exists a unique weak L^∞-solution u of the Cauchy problem for the transport equation and $u(t,x) = u_0(\phi_t^{-1}(x))$.

The main interest of this result is due to the fact that we can consider some p in the critical interval $(1,2]$ not covered by Hypothesis 2 in [8]; recall that this requires that there exists $p \in (2,+\infty)$, such that

$$\operatorname{div} b \in L^p([0,T] \times \mathbb{R}^d), \quad d \geq 2. \tag{17}$$

The next uniqueness result requires an additional hypothesis of Sobolev regularity for b (beside the usual Hölder regularity) but allows to avoid *global* integrability assumptions on $\operatorname{div} b$.

Theorem 14 *Assume $u_0 \in L^\infty(\mathbb{R}^d)$, $\operatorname{div} b \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^d)$ and*

$$b \in L^1(0,T;W^{\theta,1}_{\text{loc}}(\mathbb{R}^d)) \cap L^\infty(0,T;C^\alpha(\mathbb{R}^d)) \tag{18}$$

with $\alpha, \theta \in (0,1)$ and $\alpha + \theta > 1$. Then there exists a unique weak L^∞-solution u of the Cauchy problem for the transport equations and $u(t,x) = u_0(\phi_t^{-1}(x))$.

Remark 15 *Recall that $b \in L^1(0,T;W^{\theta,1}_{\text{loc}}(\mathbb{R}^d))$ if $\int_0^T \|b(s,\cdot)\|_{W^{\theta,1}(\mathcal{D})} ds < \infty$, for any smooth bounded domain $\mathcal{D} \subset \mathbb{R}^d$. Since $C^\alpha(\mathcal{D}) \subset W^{\theta,1}(\mathcal{D})$, for any $\theta < \alpha$, we deduce that Hypothesis 1 implies* (18) *when $\alpha > 1/2$; in particular Theorem 14 follows from Theorem 6 but only when $\alpha > 1/2$.*

The proofs of both theorems follow ideas of [8, Section 5], using the results below on the commutator and on the regularity of the Jacobian of the flow. The following commutator estimates follows from [8, Lemma 22].

Corollary 16 *Assume $v \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, $\operatorname{div} v \in L^1_{loc}(\mathbb{R}^d)$, $g \in L^\infty_{loc}(\mathbb{R}^d)$ and $\rho \in C^\infty_r(\mathbb{R}^d)$.*

(i) If there exists $\theta \in (0,1)$ such that $v \in W^{\theta,1}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$, then

$$\left| \int_{\mathbb{R}^d} \mathcal{R}_\varepsilon [g,v](x) \rho(x) dx \right| \leq C_r \|g\|_{L^\infty_{r+1}} \left(\|\rho\|_{L^\infty_r} \|\operatorname{div} v\|_{L^1_{r+1}} + [\rho]_{C^{1-\theta}_r} [v]_{W^{\theta,1}_{r+1}} \right).$$

(ii) If there exists $\alpha \in (0,1)$ such that $v \in C^\alpha_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$, then

$$\left| \int_{\mathbb{R}^d} \mathcal{R}_\varepsilon [g,v](x) \rho(x) dx \right| \leq C_r \|g\|_{L^\infty_{r+1}} \left(\|\rho\|_{L^\infty_r} \|\operatorname{div} v\|_{L^1_{r+1}} + [v]_{C^\alpha_{r+1}} [\rho]_{W^{1-\alpha,1}_r} \right).$$

Proof. We have

$$\left| \iint g(x') D_x \vartheta_\varepsilon (x-x') \left(\rho(x)-\rho(x')\right) [v(x)-v(x')]\, dx dx' \right|$$

$$\leq \frac{\varepsilon^{1-\theta}}{\varepsilon} [\rho]_{C_r^{1-\theta}} \|g\|_{L^\infty_{r+1}} \frac{1}{\varepsilon^d} \iint_{B(r+1)^2} |D_x \vartheta(\frac{x-x'}{\varepsilon})| \frac{|v(x)-v(x')|}{|x-x'|^{\theta+d}} |x-x'|^{\theta+d} dx dx'$$

$$\leq [\rho]_{C_r^{1-\theta}} \|g\|_{L^\infty_{r+1}} \|D\theta\|_\infty [v]_{W^{\theta,1}_{r+1}}$$

The second statement has a similar proof. ■

The previous result can be extended to the case in which commutators are composed with a flow.

Lemma 17 *Let ϕ be a C^1-diffeomorphism of $\mathbb{R}^d$ ($J\phi$ denotes its Jacobian). Assume $v \in L^\infty_{loc}\left(\mathbb{R}^d, \mathbb{R}^d\right)$, $\operatorname{div} v \in L^1_{loc}\left(\mathbb{R}^d\right)$, $g \in L^\infty_{loc}\left(\mathbb{R}^d\right)$.*

Then, for any $\rho \in C^\infty_r(\mathbb{R}^d)$ and any $R > 0$ such that $\operatorname{supp}(\rho \circ \phi^{-1}) \subseteq B(R)$, we have a uniform bound of $\int \mathcal{R}_\varepsilon [g, v] (\phi(x)) \rho(x)\, dx$ under one of the following conditions:

(i) there exists $\theta \in (0,1)$ such that $v \in W^{\theta,1}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d)$, $J\phi \in C^{1-\theta}_{\mathrm{loc}}(\mathbb{R}^d)$;

(ii) there exists $\alpha \in (0,1)$ such that $J\phi \in W^{1-\alpha,1}_{\mathrm{loc}}(\mathbb{R}^d)$, $v \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d)$.

Moreover, under one of the previous conditions, we also have

$$\lim_{\varepsilon \to 0} \int \mathcal{R}_\varepsilon [g, v] (\phi(x)) \rho(x)\, dx = 0.$$

Proof. By a change of variables $\int \mathcal{R}_\varepsilon[g, v](\phi(x))\rho(x) dx = \int \mathcal{R}_\varepsilon[g, v](y)\rho_\phi(y) dx$ where the function $\rho_\phi(y) = \rho(\phi^{-1}(y)) J\phi^{-1}(y)$ has the support strictly contained in the ball of radius R. Clearly, $\|\rho_\phi\|_{L^\infty_R} \leq \|\rho\|_{L^\infty_r} \|J\phi^{-1}\|_{L^\infty_R}$. To prove the result, we have to check that Corollary 16 can be applied with ρ_ϕ instead of ρ.

(i) To apply Corollary 16 (i), we need to check that $\rho_\phi \in C^{1-\theta}_{loc}$. This follows since

$$[\rho_\phi]_{C_R^{1-\theta}} \leq \|J\phi^{-1}\|_{L^\infty_R} [\rho(\phi^{-1}(\cdot))]_{C_R^{1-\theta}} + \|\rho\|_{L^\infty_r} [J\phi^{-1}]_{C_R^{1-\theta}}$$
$$\leq \|D\phi^{-1}\|_{L^\infty_R} \|D\rho\|_{L^\infty_r} [D\phi^{-1}]_{C_R^{1-\theta}} + \|\rho\|_{L^\infty_r} [D\phi^{-1}]_{C_R^{1-\theta}}.$$

and the bound follows.

(ii) To apply Corollary 16 (ii), we need to check that $\rho_\phi \in W^{1-\alpha,1}_{loc}$: first

$$[\rho_\phi]_{W_R^{1-\alpha,1}} \leq \|J\phi^{-1}\|_{L^\infty_R} [\rho \circ \phi^{-1}]_{W_R^{1-\alpha,1}} + [J\phi^{-1}]_{W_R^{1-\alpha,1}} \|\rho\|_{L^\infty_r}$$

and since

$$[\rho \circ \phi^{-1}]_{W_R^{1-\alpha,1}} \leq \|D(\rho \circ \phi^{-1})\|_{L^1_R} \leq \|D\rho\|_{L^1_r} \|D\phi^{-1}\|_{L^\infty_R}$$

we find

$$[\rho_\phi]_{W_R^{1-\alpha,1}} \leq C_R \|D\rho\|_{L^1_r} \|D\phi^{-1}\|_{L^\infty_R} \|J\phi^{-1}\|_{L^\infty_R} + [J\phi^{-1}]_{W_R^{1-\alpha,1}} \|\rho\|_{L^\infty_r}$$

and the bound follows. ■

Finally the next theorem extends the analysis of the Jacobian of the flow presented in Section 2 and links the regularity condition on $J\phi$ required in Lemma 17 (ii) to the assumption on the divergence of b stated in Theorem 13.

Theorem 18 *Let $d \geq 2$. Assume Hypothesis 1 and the existence of $p \in (\frac{2d}{d+2\alpha}, 2]$ and $q > 2$ such that $\operatorname{div} b \in L^q(0,T;L^p(\mathbb{R}^d))$. Then, for any $r > 0$, $J\phi \in L^p(0,T;W_r^{1-\alpha,p})$, P-a.s.*

Proof. In the sequel we assume $\sigma = 1$ to simplify notation.

The first part of the proof is similar to the one of [8, Theorem 11]. Indeed Step 1 can be carried on thanks to the chain rule for fractional Sobolev spaces: if $f : \mathbb{R}^d \to \mathbb{R}$ is a continuous function, of class $W_{loc}^{1-\alpha,p}(\mathbb{R}^d)$ and $g : \mathbb{R} \to \mathbb{R}$ is a C^∞ function, then $g \circ f \in W_{loc}^{1-\alpha,p}(\mathbb{R}^d)$ and

$$[(g \circ f)]^p_{W_r^{1-\alpha,p}} \leq \left(\sup_{x \in B(r)} |g'(f(x))| \right)^p [f]^p_{W_r^{1-\alpha,p}},$$

for every $r > 0$. The modification of Step 2 does not pose any problem, so we only consider the last steps of the proof.

Step 3. To prove the assertion it is enough to check that the family $(\psi_\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(\Omega \times (0,T); W_r^{1-\alpha,p})$.

Indeed, once we have proved this fact, we can extract from the previous sequence ψ_{ε_n} a subsequence which converges weakly in $L^p(\Omega \times (0,T); W_r^{1-\alpha,p})$ to some γ. This in particular implies that such subsequence converges weakly in $L^p(\Omega \times (0,T), L_r^p)$ to γ so we must have that $\gamma = J\phi$.

We introduce the following Cauchy problem, for $\varepsilon \geq 0$,

$$\begin{cases} \dfrac{\partial F^\varepsilon}{\partial t} + \dfrac{1}{2}\Delta F^\varepsilon + DF^\varepsilon \cdot b^\varepsilon = \operatorname{div} b^\varepsilon, \quad t \in [0,T[\\ \qquad\qquad\qquad\qquad F^\varepsilon(T,x) = 0, \quad x \in \mathbb{R}^d. \end{cases} \tag{19}$$

This problem has a unique solution F^ε in the space $L^q(0,T;W^{2,p}(\mathbb{R}^d)$. Moreover, there exists a positive constant $C = C(p,q,d,T,\|b\|_\infty)$ such that

$$\|F^\varepsilon\|_{L^q(0,T;W^{2,p}(\mathbb{R}^d))} \leq C \|\operatorname{div} b\|_{L^q(0,T;L^p(\mathbb{R}^d))}, \tag{20}$$

for any $\varepsilon \geq 0$. This result can be proved by using [13, Theorem 1.2] and repeating the argument of the proof in [14, Theorem 10.3]. This argument works without difficulties in the present case in which b (and so b^ε) is globally bounded and $\operatorname{div} b \in L^q(0,T;L^p(\mathbb{R}^d))$ with $p,q \in (1,+\infty)$.

From the previous result we can also deduce, since we are assuming $q > 2$, that $F^\varepsilon \in C([0,T];W^{1,p}(\mathbb{R}^d))$, for any $\varepsilon \geq 0$, and moreover there exists a positive constant $C = C(p,q\ d,T,\|b\|_\infty)$ such that

$$\sup_{t \in [0,T]} \|F^\varepsilon(t,\cdot)\|_{W^{1,p}(\mathbb{R}^d)} \leq C \|\operatorname{div} b\|_{L^q(0,T;L^p(\mathbb{R}^d))}. \tag{21}$$

We only give a sketch of proof of (21). Define $u^\varepsilon(t,x) = F^\varepsilon(T-t,x)$; we have the explicit formula

$$u^\varepsilon(t,x) = \int_0^t P_{t-s} g^\varepsilon(s,\cdot)(x) ds,$$

where (P_t) is the heat semigroup and $g^\varepsilon(t,x) = Du^\varepsilon(t,x)\cdot b^\varepsilon(T-t,x) - \operatorname{div} b^\varepsilon(T-t,x)$. We get, since $q>2$ and $q' = \frac{q}{q-1} < 2$,

$$\begin{aligned}
\|D_x u^\varepsilon(t,\cdot)\|_{L^p} &\le c \int_0^t \frac{1}{(t-s)^{1/2}} \|g^\varepsilon(s,\cdot)\|_{L^p} ds \\
&\le C \Big(\int_0^T \frac{1}{s^{q'/2}} ds \Big)^{1/q'} \Big(\int_0^T \|\operatorname{div} b(s,\cdot)\|_{L^p}^q ds \Big)^{1/q}
\end{aligned}$$

and so (21) holds. Using Itô formula we find (remark that $F^\varepsilon(t,\cdot) \in C_b^2(\mathbb{R}^d)$)

$$F^\varepsilon(t,\phi_t^\varepsilon(x)) - F^\varepsilon(0,x) - \int_0^t DF^\varepsilon(s,\phi_s^\varepsilon(x))\cdot dW_s = \int_0^t \operatorname{div} b^\varepsilon(s,\phi_s^\varepsilon(x))\, ds = \psi_\varepsilon(t,x). \tag{22}$$

Since we already know that $(\psi_\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(\Omega\times(0,T), L^p_r)$ and since $p\le 2$, to verify that $(\psi_\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(\Omega\times(0,T); W_r^{1-\alpha,p})$, it is enough to prove that $E\int_0^T [\psi_\varepsilon(t,\cdot)]^2_{W_r^{1-\alpha,2}} dt \le C$, for any $\varepsilon>0$. We give details only for the most difficult term $\int_0^t DF^\varepsilon(s,\phi_s^\varepsilon(x)) dW_s$ in (22). The $F(0,x)$ term can be controlled using (21) and the others are of easier estimation. We show that there exists a constant $C>0$ (independent on ε) such that

$$E\int_0^T dt \left[\int_0^t DF^\varepsilon(s,\phi_s^\varepsilon(\cdot))\, dW_s \right]^2_{W_r^{1-\alpha,2}} \le C \tag{23}$$

We have

$$\begin{aligned}
&E\left[\int_0^T dt \int_{B(r)} \int_{B(r)} \frac{|\int_0^t (DF^\varepsilon(s,\phi_s^\varepsilon(x)) - DF^\varepsilon(s,\phi_s^\varepsilon(x'))) dW_s|^2}{|x-x'|^{(1-\alpha)2+d}} dx\, dx' \right] \\
&= \int_0^T \int_{B(r)} \int_{B(r)} E \int_0^t \frac{|DF^\varepsilon(s,\phi_s^\varepsilon(x)) - DF^\varepsilon(s,\phi_s^\varepsilon(x'))|^2}{|x-x'|^{(1-\alpha)2+d}} ds\, dx\, dx', \\
&= E\int_0^T dt \int_0^t ds \int_{B(r)} \int_{B(r)} \frac{|DF^\varepsilon(s,\phi_s^\varepsilon(x)) - DF^\varepsilon(s,\phi_s^\varepsilon(x'))|^2}{|x-x'|^{(1-\alpha)2+d}} dx\, dx' \\
&\le TE\left[\int_0^T ds \int_{B(r)} \int_{B(r)} \frac{|DF^\varepsilon(s,\phi_s^\varepsilon(x)) - DF^\varepsilon(s,\phi_s^\varepsilon(x'))|^2}{|x-x'|^{(1-\alpha)2+d}} dx\, dx' \right], \\
&\le TE\int_0^T [DF^\varepsilon(s,\phi_s^\varepsilon(\cdot))]^2_{W_r^{1-\alpha,2}}\, ds
\end{aligned}$$

By the Sobolev embedding the $W_r^{1-\alpha,2}$-seminorm can be controlled by the norm in $W_r^{1,p}$ if

$$1-\frac{d}{p}\geq(1-\alpha)-\frac{d}{2}.$$

This holds if $p\geq\frac{2d}{d+2\alpha}$. Then we consider p_1 such that $p>p_1>\frac{2d}{d+2\alpha}$ and show that

$$E\int_0^T\|DF^\varepsilon(s,\phi_s^\varepsilon(\cdot))\|^2_{W_r^{1,p_1}}ds\leq C<\infty, \tag{24}$$

where C is independent on ε.

Step 4. To obtain (24) we estimate

$$E\int_0^T ds\Big(\int_{B(r)}|D^2F^\varepsilon\left(s,\phi_s^\varepsilon\left(x\right)\right)D\phi_s^\varepsilon\left(x\right)|^{p_1}dx\Big)^{\frac{2}{p_1}}$$

A similar term has been already estimated in the proof of Theorem 11 in [8]. Since

$$\int_{B(r)}\left(\int_0^T E\left[\left|D\phi_s^\varepsilon\left(x\right)\right|^r\right]ds\right)^\gamma dx<\infty,$$

for every $r,\gamma\geq1$ (see (8)), by the Hölder inequality, it is sufficient to prove that

$$\int_0^T E\left[\left(\int_{B(r)}\left|D^2F^\varepsilon\left(s,\phi_s^\varepsilon\left(x\right)\right)\right|^p dx\right)^{\frac{2}{p}}\right]dt\leq C<\infty.$$

We have

$$\begin{aligned}
&\int_0^T E\left[\left(\int_{B(r)}\left|D^2F^\varepsilon\left(s,\phi_s^\varepsilon\left(x\right)\right)\right|^p dx\right)^{\frac{2}{p}}\right]dt\\
&=E\left[\int_0^T ds\left(\int_{\phi_s^\varepsilon(B(r))}\left|D^2F^\varepsilon\left(s,y\right)\right|^p J(\phi_s^\varepsilon)^{-1}(y)dy\right)^{\frac{2}{p}}\right]\\
&\leq\sup_{s\in[0,T],\,y\in\mathbb{R}^d}E[J(\phi_s^\varepsilon)^{-1}\left(y\right)]^{2/p}\int_0^T\left(\int_{\mathbb{R}^d}\left|D^2F^\varepsilon\left(s,y\right)\right|^p dy\right)^{\frac{2}{p}}\leq C<\infty,
\end{aligned}$$

where, using the results of [8, Section 3] and the bound (20), C is independent on $\varepsilon>0$. The proof is complete. ■

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